



The phase structure of Einstein–Cartan theory

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ABSTRACT

In the Einstein–Cartan theory of torsion-free gravity coupling to massless fermions, the four-fermion interaction is induced and its strength is a function of the gravitational and gauge couplings, as well as the Immirzi parameter. We study the dynamics of the four-fermion interaction to determine whether effective bilinear terms of massive fermion fields are generated. Calculating one-particle-irreducible two-point functions of fermion fields, we identify three different phases and two critical points for phase transitions characterized by the strength of four-fermion interaction: (1) chiral symmetric phase for massive fermions in strong coupling regime; (2) chiral symmetric broken phase for massive fermions in intermediate coupling regime; (3) chiral symmetric phase for massless fermions in weak coupling regime. We discuss the scaling-invariant region for an effective theory of massive fermions coupled to torsion-free gravity in the *low-energy limit*.

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1. Introduction

The self-dual connection of a Yang–Mills gauge theory introduced in the Ashtekar formalism for General Relativity [1] is crucial for the canonical quantization procedure, leading to the non-perturbative quantum theory of gravity, *Loop Quantum Gravity* [2–4]. The complex Ashtekar's connection with reality condition and the real Barbero real connection [5] are linked by a canonical transformation of the connection with the Immirzi parameter $\gamma \neq 0$ [6], which has crucial effects on quantum gravity [7] at the Planck energy scale, but does not affect the classical dynamics of torsion-free gravity. However, when fermion fields are present and coupled to gravity, yielding a non-vanishing torsion tensor [8], and the Einstein–Cartan theory for torsion-free gravity coupling to fermions should be modified. Indeed, Ref. [9] show that the four-fermion interacting strength in the Einstein–Cartan theory is related to the Immirzi parameter, which can possibly lead to physical effects observable. Thus, it is worthwhile to study the dynamics of these quadrilinear terms of fermion fields in terms of the four-fermion interacting strength to see whether effective bilinear terms of massive fermions are generated.

2. Einstein–Cartan theory

In the Palatini framework, the basic gravitational variables constitute a pair of tetrad and spin-connection fields $(e_\mu^a, \omega_\mu^{ab})$. They are 1-form fields on \mathcal{M} the 4-dimensional Euclidean space-time manifold, taking values, respectively, in the vector space V and in

the Lie algebra $so(\eta)$ of the group $SO(\eta)$ of the linear transformations of V preserving $\eta^{ab} = (+, +, +, +)$. The 2-form curvature associating with the spin-connection is

$$R^{ab} = d\omega^{ab} - \omega^{ae} \wedge \omega^b_e. \quad (1)$$

The Palatini action for gravitational field is given by,

$$S_P(e, \omega) = \frac{1}{4k} \int_{\mathcal{M}} d^4x \det(e) \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}, \quad (2)$$

where $k \equiv 8\pi G$. The relationship between spin-connection ω_μ^{ab} and the tetrad e_μ^a is determined by $\delta S_P(e, \omega)/\delta \omega = 0$, Cartan's structure equation,

$$de^a - \omega^{ab} \wedge e_b = 0, \quad (3)$$

which gives the torsion-free spin-connection: $\omega = \omega(e)$. Replacing ω in Eq. (2) by $\omega = \omega(e)$, the Palatini action $S_P[e, \omega(e)]$ reduces to the Einstein–Hilbert action and its variation with respect to the tetrad field e_μ^a leads to the Einstein field equation,

$$\epsilon_{abcd} e^b \wedge R^{cd}[\omega(e)] = 0. \quad (4)$$

Adding the Host modification with the Immirzi parameter γ , one has

$$S_H(e, \omega) = S_P(e, \omega) - \frac{1}{2k\gamma} \int_{\mathcal{M}} d^4x \det(e) e_a \wedge e_b \wedge R^{ab}. \quad (5)$$

Introducing massless Dirac fermions ψ coupled to the gravitational field described by $(e_\mu^a, \omega_\mu^{ab})$, we adopt the fermion action of Ashtekar–Romano–Tate type [10],

$$S_F(e, \omega, \psi, \bar{\psi}) = \frac{1}{2} \int_{\mathcal{M}} d^4x \det(e) [\bar{\psi} e^\mu \mathcal{D}_\mu \psi + \text{h.c.}], \quad (6)$$

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where the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{4}\beta\omega_\mu, \quad (7)$$

β is the gauge coupling between fermion and spin-connection fields, Dirac-matrix valued tetrad and spin-connection fields are $e^\mu \equiv e^\mu_a \gamma^a$ and $\omega_\mu \equiv \omega_\mu^{ab} \sigma_{ab}$. The anti-hermitian Dirac matrix $\gamma_a^\dagger = -\gamma_a$, $\gamma_a^2 = -1$ ($a = 0, 1, 2, 3$), $\{\gamma_a, \gamma_b\} = -2\eta_{ab}$, and the hermitian spinor matrix $\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$. The actions (2), (6) are invariant under the diffeomorphisms of the manifold \mathcal{M} , and can be separated into left- and right-handed parts [11], with respect to local $SU_L(2)$ - and $SU_R(2)$ -Lorentz symmetries.

The Palatini action (2) and fermion action (6) give the Einstein–Cartan action,

$$S_{EC} = S_P(e, \omega) + S_F(e, \omega, \psi, \bar{\psi}). \quad (8)$$

Analogously to Eq. (3), $\delta S_{EC}(e, \omega)/\delta\omega = 0$ gives Cartan’s structure equation,

$$de^a - \omega^{ab} \wedge e_b - T^a = 0, \quad (9)$$

where the non-vanishing torsion field $T^a = k\beta e_b \wedge e_c J^{ab,c}$, relating to the fermion spin-current

$$J^{ab,c} = \frac{i}{4}\bar{\psi}\{\sigma^{ab}, \gamma^c\}\psi = \frac{1}{4}\epsilon^{abcd}\bar{\psi}\gamma_d\gamma^5\psi, \quad (10)$$

and $\{\sigma^{ab}, \gamma^c\} = i\epsilon^{abcd}\gamma^5\gamma_d$. The hermitian γ_5 -matrix $\gamma_5 = \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma_5^\dagger = \gamma_5$ and $\gamma_5^2 = 1$. The totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{abcd}e_\mu^a e_\nu^b e_\rho^c e_\sigma^d$. The solution to Eq. (9) is

$$\omega_\mu^{ab} = \omega_\mu^{ab}(e) + \tilde{\omega}_\mu^{ab}, \quad \tilde{\omega}_\mu^{ab} = k\beta e_\mu^c J^{ab}{}_c, \quad (11)$$

where the connection $\omega_\mu^{ab}(e)$ obeys Eq. (3) for torsion-free case. The fermion spin-current (10) contributes only to the pseudo-trace axial vector of torsion tensor, which is one of irreducible parts of torsion tensor [12]. Replacing the spin-connection ω in the Einstein–Cartan action (8) by (11),

$$S_P[e, \omega, \psi, \bar{\psi}] \rightarrow S_P[e, \omega(e)] - \frac{1}{16}k\beta^2 \int_{\mathcal{M}} d^4x \times \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi), \quad (12)$$

$$S_F[e, \omega, \psi, \bar{\psi}] \rightarrow S_F[e, \omega(e), \psi, \bar{\psi}] - \frac{2}{16}k\beta^2 \int_{\mathcal{M}} d^4x \times \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi), \quad (13)$$

one obtains the well-known Einstein–Cartan theory: the standard tetrad action of torsion-free gravity coupling to fermions,

$$\begin{aligned} S_{EC}[e, \omega(e), \psi, \bar{\psi}] \\ = S_P[e, \omega(e)] + S_F[e, \omega(e), \psi, \bar{\psi}] \\ - \frac{3}{16}k\beta^2 \int_{\mathcal{M}} d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi). \end{aligned} \quad (14)$$

In the case of the Host action (5), the four-fermion interaction term is given by [9]

$$-\frac{3}{16} \frac{\gamma^2}{\gamma^2 + 1} k\beta^2 \int_{\mathcal{M}} d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi). \quad (15)$$

As we can see from Eqs. (6)–(14), the bilinear term (6) of massless fermion fields coupled to the spin-connection (7) is bound to yield a non-vanishing torsion field T^a (9), which is local and static. As a result, the spin-connection ω is no longer torsion-free

and acquires a torsion-related spin-connection $\tilde{\omega}_\mu^{ab}$ (11), in addition to the torsion-free spin-connection $\omega_\mu^{ab}(e)$. The torsion-related spin-connection $\tilde{\omega}_\mu^{ab}$ is related to the fermion spin-current (10). The quadratic term of the spin-connection field ω in Eq. (1) and the coupling between the spin-connection field and fermion spin-current in Eqs. (6), (7) lead to the quadrilinear terms of fermion fields in Eqs. (12) and (13). Another way to see this is to treat the static torsion-related spin-connection $\tilde{\omega}_\mu^{ab}$ (11) as a static auxiliary field, which has its quadratic term and linear coupling to the spin-current of fermion fields. Performing the Gaussian integral of the static auxiliary field, we exactly obtain the quadrilinear term (14), in addition to the torsion-free action.

3. A postulation and fermion-mass generation

The gauge principle requires the action of gravitational and fermion fields be invariant under the diffeomorphisms of the manifold \mathcal{M} and local Lorentz transformations. This leads to a pseudo-trace axial vector of static (non-dynamics) torsion field, which is related to the spin-current of fermion fields. The interaction between the pseudo-trace axial vector of torsion fields and the spin-current of fermion fields results in the four-fermion interaction (quadrilinear terms in massless fermion fields). As a consequence, the gauge-invariant action consists of the torsion-free action of gravitational and fermion fields and four-fermion interaction. We thus postulate that it is impossible to have any gauge-invariant theories made by the bilinear terms of massless fermion fields coupled to torsion-free gravitational field, and quadrilinear terms (or high-order terms) of massless fermion fields must be present [13].

The quadrilinear term (15) is a dimension-6 operator, and four-fermion coupling is in terms of the Immirzi parameter γ , gravitational-coupling k and gauge-coupling β . If quantum gravity is taken into account, we expect non-local and high-dimensional operators ($d > 6$), which contain high-order derivatives. In this case, the torsion-related spin-connection $\tilde{\omega}_\mu^{ab}$ (11) is not completely static, rather has a mass of the order of the Planck mass, mediating in a few Planck length to form effective high-dimensional operators of massless fermion fields. The fundamental fields e , ω , ψ , and operators $\mathcal{O}(e, \omega, \psi, k, \beta)$ are functions of the couplings k , β , γ , depending on the energy-scale \mathcal{E} . First we should adopt appropriate vacuum expectational value (e.v.e.) of operators $\langle \mathcal{O}(e, \omega, \psi, k, \beta) \rangle$ as order parameters, to describe different phases and phase transitions in the space of couplings k , β , and parameter γ . Second, we try to identify the scaling-invariant regime (ultraviolet fix points) for the low-energy limit ($\mathcal{E}/m_p \rightarrow 0$), where the variation of fundamental fields, couplings and operators as functions of the energy-scale \mathcal{E} is govern by renormalization group equations. Third, in such scaling-invariant regime we try to determine the relevant and renormalizable operators that are effective dimension-4 operators, to obtain an effective low-energy theory for the present Universe.

In this Letter, we are interested in the one-particle-irreducible (1PI) two-point functions of fermion fields $\langle \psi(0)\bar{\psi}(x) \rangle$, since they contribute to effective operators for the energy-momentum tensor entering the right-hand side of the Einstein equation (4) for classical gravity. Our goal is limited to find non-trivial fermion-mass operators ($\langle \psi\bar{\psi} \rangle \neq 0$) in terms of the four-fermion interacting strength. For convenience in calculations, using the Planck mass m_p , we rescale fermion fields $\psi \rightarrow \psi/m_p$ and rewrite four-fermion interaction (15) as

$$\begin{aligned} g \int_{\mathcal{M}} d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi), \\ g = \frac{3}{16} \frac{\gamma^2}{\gamma^2 + 1} k\beta^2 m_p^4, \end{aligned} \quad (16)$$

where the four-fermion coupling g has dimension $[m_p^2]$. We assume the gauge-coupling β to be perturbatively small.

4. Weak four-fermion coupling

In the weak-coupling limit $g/m_p^2 \ll 1$, the dimension-3 fermion-mass operators $\langle \bar{\psi} \psi \rangle$ identically vanish ($\langle \bar{\psi} \psi \rangle \equiv 0$), the action (14) gives a weakly interacting, massless $SU_L(2) \otimes SU_R(2)$ fermion spectrum. We define this as the “weak-coupling symmetric phase”. In the intermediate range of coupling g , there is a “broken phase” where spontaneous symmetry breaking occurs. Using large- N_f expansion technique¹ shows that the four-fermion interaction (14) undergoes Nambu–Jona-Lasinio (NJL) spontaneous chiral-symmetry breaking [14]. In this symmetry broken phase, $SU_L(2) \otimes SU_R(2)$ chiral symmetry is violated by non-vanishing mass-operators

$$\frac{1}{2} \Sigma(p) = g \int d^4x \det(e) e^{-ipx} \langle \bar{\psi}(0) \cdot \psi(x) \rangle \neq 0, \quad (17)$$

where $\langle \cdots \rangle$ is the average with respect to the partition function Z of fermionic part of the action (14)

$$\langle \cdots \rangle = \frac{1}{Z} \int d\bar{\psi} d\psi (\cdots) \exp\{-S_{\text{EC}}[e, \omega(e), \bar{\psi}, \psi]\}. \quad (18)$$

The non-vanishing mass operator (17) obeys the NJL gap-equation,

$$\Sigma(p) = \tilde{g} \int \frac{d^4q}{(2\pi)^4} \frac{\Sigma(q)}{q^2 + (\Sigma(q)/m_p)^2}, \quad (19)$$

where momentum q and coupling $\tilde{g} = gN_f/m_p^2$ are dimensionless. The critical point $\tilde{g}_c = 8\pi^2$, which can be obtained by $\Sigma \rightarrow 0^+$, separates the “broken phase” ($\Sigma \neq 0$, $\tilde{g} > \tilde{g}_c$) from the “weak-coupling symmetric phase” ($\Sigma \equiv 0$, $\tilde{g} < \tilde{g}_c$). $\Sigma(p) \sim \mathcal{O}(m_p)$ for $\tilde{g} > \tilde{g}_c$. The inverse propagators of these fermions can then be written as,

$$S^{-1}(p) = i\gamma_\mu p^\mu + \Sigma(p). \quad (20)$$

The $SU_L(2) \otimes SU_R(2)$ chiral symmetry is realized to be $SU(2)$ with three Goldstone modes and a massive Higgs mode that are not presented here. Eq. (20) corresponds to the bilinear term of massive fermion fields in the effective action, which does not preserve chiral symmetries.

5. Strong four-fermion coupling

We turn to the strong-coupling region, where four-fermion coupling g in (14) is sufficiently larger than a certain critical value g_{crit} , bound states of three fermions (three-fermion states) are formed

$$\Psi = \frac{m_p}{2} (\bar{\psi} \cdot \psi) \psi, \quad (21)$$

which can be understood as a bound state of one fermion and one composite boson $(\bar{\psi} \cdot \psi)$. These three-fermion states (21) carry the appropriate quantum numbers of the gauge group that accommodates ψ . The fermion-mass operator is $\bar{\psi} \Psi$ and thus massive fermion spectrum is consistent with the chiral symmetry $SU_L(2) \otimes SU_R(2)$.

For the purpose of understanding three-fermion states and their spectra, we henceforth focus on the strong-coupling region ($g/m_p^2 \gg 1$). We make a rescaling of fermion fields,

$$\psi(x) \rightarrow g^{1/4} \psi(x), \quad (22)$$

and rewrite the fermion action in terms of the new fermion fields

$$S_f(x) = \frac{1}{2g^{1/2}} [\bar{\psi}(x) \gamma_\mu \partial^\mu \psi(x) + \text{h.c.}], \quad (23)$$

$$S_i(x) = (\bar{\psi} \gamma^d \gamma^5 \psi) (\bar{\psi} \gamma_d \gamma^5 \psi), \quad (24)$$

where the gauge coupling β is assumed to be weak. For the limit of strong coupling $g/m_p^2 \rightarrow \infty$, the kinetic terms $S_f(x)$ can be dropped and we calculate the partition function Z (18) in this strong-coupling limit. With $S_i(x)$ given in Eq. (24), the integral of $e^{-S_i(x)}$ is calculated by Grassmann anticommuting algebra,

$$Z = \Pi_x \int [d\bar{\psi}(x) d\psi(x)] \exp[-S_i(x)] = \Pi_x 2^4 \neq 0, \quad (25)$$

which shows a non-trivial strong-coupling limit. About this strong-coupling limit (25), we now can perform the strong-coupling expansion of $e^{-S_f(x)}$ in powers of $1/g$ to calculate Green-functions of fermion fields $\langle \psi(x_1) \bar{\psi}(x_2) \cdots \psi(x_n) \rangle$ [15,16]. In order to do integral of Grassmann anticommuting algebra, we rewrite the kinetic term $S_f(x)$ (23) as a hopping term in the Planck spacing a^μ , $|a^\mu| = a = 1/m_p$,

$$S_f(x) = \frac{1}{2g^{1/2}a} [\bar{\psi}(x) \gamma_\mu \psi(x + a^\mu) - \bar{\psi}(x + a^\mu) \gamma_\mu \psi(x)]. \quad (26)$$

We consider the following two-point functions that form the propagator of the composite Dirac particle

$$S_{LL}(x) \equiv \langle \psi(0), \bar{\psi}(x) \rangle, \quad (27)$$

$$S_{ML}(x) \equiv (2a) \langle \psi(0), \bar{\Psi}(x) \rangle, \quad (28)$$

$$S_{MM}(x) \equiv (2a)^2 \langle \Psi(0), \Psi(x) \rangle. \quad (29)$$

In the lowest non-trivial order $\mathcal{O}(1/g)$, we obtain the following recursion relations²

$$S_{LL}(x) = \frac{1}{g} \left(\frac{1}{2a} \right)^\dagger \sum_\mu S_{ML}(x + a^\mu) \gamma_\mu, \quad (30)$$

$$S_{ML}(x) = \frac{\delta(x)}{2g} + \frac{1}{g} \left(\frac{1}{2a} \right)^\dagger \sum_\mu S_{LL}(x + a^\mu) \gamma_\mu, \quad (31)$$

$$S_{MM}(x) = \frac{1}{g} \left(\frac{1}{2a} \right)^\dagger \sum_\mu \gamma_\mu \gamma_0 S_{ML}^\dagger(x + a^\mu) \gamma_0, \quad (32)$$

where for an arbitrary function $f(x)$,

$$\sum_\mu^\dagger f(x) = \sum_\mu [f(x + a^\mu) - f(x - a^\mu)]. \quad (33)$$

Transforming these two-point functions (27)–(29) into momentum space,

$$S_X(p) = \int d^4x e^{-ipx} S_X(x), \quad (34)$$

where $X = LL, ML, MM$, respectively, we obtain three recursion relations in momentum space

$$S_{LL}(p) = \frac{1}{g} \left(\frac{i}{4a^3} \right) \sum_\mu \sin(p^\mu a) S_{ML}(p) \gamma_\mu, \quad (35)$$

$$S_{ML}(p) = \frac{1}{2g} + \frac{i}{ga} \sum_\mu \sin(p^\mu a) S_{LL}(p) \gamma_\mu. \quad (36)$$

$$S_{MM}(p) = \frac{1}{g} \left(\frac{i}{a} \right) \sum_\mu \sin(p^\mu a) \gamma_\mu \gamma_0 S_{ML}^\dagger(p) \gamma_0. \quad (37)$$

¹ $g \ll 1$, $N_f \gg 1$ and gN_f fixed, N_f is the number of fermion flavors.

² The detailed technique and formula for calculations can be found in appendices of Ref. [16].

We solve these recursion relations (35)–(37) and obtain

$$S_{LL}(p) = \frac{\frac{i}{2a} \sum_{\mu} \sin(p^{\mu}a) \gamma_{\mu}}{\frac{1}{a^2} \sum_{\mu} \sin^2(p_{\mu}a) + M^2}, \quad (38)$$

$$\frac{1}{2a} S_{ML}(p) = \frac{\frac{1}{2} M(p)}{\frac{1}{a^2} \sum_{\mu} \sin^2(p_{\mu}a) + M^2}, \quad (39)$$

$$\left(\frac{1}{2a}\right)^2 S_{MM}(p) = \frac{\frac{i}{2a} \sum_{\mu} \sin(p^{\mu}a) \gamma_{\mu}}{\frac{1}{a^2} \sum_{\mu} \sin^2(p_{\mu}a) + M^2}, \quad (40)$$

where the chiral-invariant mass is

$$M = 2ga. \quad (41)$$

In addition, the two-point function,

$$\langle \psi(x), \bar{\psi}(0) \rangle = \frac{1}{2a} \gamma_0 S_{ML}^{\dagger}(x) \gamma_0. \quad (42)$$

As a result, in the lowest non-trivial order of the strong-coupling expansion we obtain the massive propagator of the composite Dirac fermions,

$$S(p) = \frac{\frac{i}{a} \sum_{\mu} \sin(p^{\mu}a) \gamma_{\mu} + M}{\frac{1}{a^2} \sum_{\mu} \sin^2(p_{\mu}a) + M^2} \simeq \frac{ip^{\mu} \gamma_{\mu} + M}{p^2 + M^2}, \quad (43)$$

for modes $p^{\mu}a \ll 1$. Eq. (43) corresponds to the bilinear term of massive fermion fields preserving chiral symmetries in the effective action, which can be written as,

$$S_F^{\text{eff}}(e, \omega, \Psi, \bar{\Psi}) = \frac{1}{2} \int_{\mathcal{M}} d^4x \det(e) [\bar{\Psi} e^{\mu} \mathcal{D}_{\mu} \Psi + M \bar{\Psi} \Psi] + \text{h.c.}, \quad (44)$$

and its variation with respect to the tetrad field e_{μ}^a gives rise to the energy-momentum tensor that contributes to the right-hand-side of the Einstein equation (4). This is the “strong-coupling symmetric phase”, where fermion fields are massive.

The critical value g_{crit} that separates the “strong-coupling symmetric phase” from the “broken phase” can be qualitatively determined by considering the complex composite scalar field,

$$\mathcal{A} = \bar{\psi} \cdot \psi, \quad (45)$$

and its propagator, i.e., the two-point function:

$$G(x) = \langle \mathcal{A}(0), \mathcal{A}^{\dagger}(x) \rangle. \quad (46)$$

Analogously, using the strong-coupling expansion in powers of $1/g$ ($g/m_p^2 \gg 1$), we obtain the following recursion relation in the lowest order,

$$G(x) = \frac{\delta(x)}{g} + \frac{1}{g} \left(\frac{1}{2a}\right)^2 \sum_{\pm\mu} G(x + a^{\mu}). \quad (47)$$

Going to momentum space,

$$G(q) = \int d^4x e^{-iqx} G(x), \quad (48)$$

where q is the momentum of the composite scalar \mathcal{A} , we obtain the recursion relation (47) in momentum space

$$G(q) = \frac{1}{g} + \left(\frac{1}{2a^2}\right) \frac{1}{g} \sum_{\pm\mu} \cos(q_{\mu}a) G(q). \quad (49)$$

As a result, we find the propagator of the massive composite scalar field \mathcal{A} ,

$$G(q) = \frac{4}{\frac{4}{a^2} \sum_{\mu} \sin^2\left(\frac{q_{\mu}a}{2}\right) + \mu^2} \simeq \frac{4}{q^2 + \mu^2}, \quad (50)$$

$$\mu^2 = 4\left(g - \frac{2}{a^2}\right), \quad (51)$$

where the factor 4 is due to the four components of the composite scalar field \mathcal{A} . Thus, $\mu^2 \mathcal{A} \mathcal{A}^{\dagger}$ gives the mass term of the composite scalar field \mathcal{A} in the effective Lagrangian. We assume that the 1PI vertex $\mathcal{A} \mathcal{A}^{\dagger} \mathcal{A} \mathcal{A}^{\dagger}$ is positive and the energy of ground states of the theory is bound from the below. Then, we can qualitatively discuss the second order phase transition (threshold) from the “strong-coupling symmetric phase” to the “broken phase” by examining the mass term of these composite scalars $\mu^2 \mathcal{A} \mathcal{A}^{\dagger}$. Spontaneous symmetry breaking $SU(2) \rightarrow U(1)$ occurs, where $\mu^2 > 0$ turns to $\mu^2 < 0$. Eq. (51) for $\mu^2 = 0$ gives rise to the critical value g_{crit} :

$$g_{\text{crit}} a^2 = 2, \quad (52)$$

where a phase transition takes place between the “strong-coupling symmetric phase” and the “broken phase”.

6. Some discussions

As already mentioned, high-dimensional operators of massless fermion fields containing high-order derivatives are expected if the quantum gravity is included. In this case the four-fermion coupling (16) and fermion-mass (41) should be functions of fermion's momentum p^{μ} . Both the phase-structure and critical points for phase-transition characterized by the coupling g (16) depend clearly also on the Immirzi parameter γ . Although three different phases have been differentiated, we have not been able to identify the scaling-invariant region for the *low-energy limit* where some of high-dimensional operators receive anomalous dimensions become relevant operators of effective dimension-4, others are non-relevant and suppressed. We expect that the scaling-invariant regime be probably near to the critical point (52) so that the low-energy effective theory preserves chiral-gauge symmetry in high-energies and has a soft symmetry-breaking for fermion masses in low-energies. In this Letter we discuss the phase-structure of the Einstein–Cartan theory and a theoretical possibility to understand how fermion fields become massive and couple to torsion-less gravitational field.

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